Differential equations

Reference 21.1, 21.5 – 21.7 K. Sydsaeter and P. Hammond

Introduction

Many economic models with a temporal dimension involve relationships between the rate of change of a variable and its value at a given point in time. For example, a model of price dynamics might assume that rate of change of a price is proportional to the difference between the demand and supply at that price, and a macroeconomic model of economic growth might assume that the rate of change of the capital stock is a constant fraction of the value of output. When time is modeled as a continuous variable, they may be modeled as differential equations. [when time is discrete, we have difference equations]

We will only be discussing the first order ordinary differential equation

which is a relationship between a variable t, the value of a function x of a single variable at t, and the first derivative of x at t.

$$F(t, x, \dot{x}) = 0 \quad \text{where } \dot{x} = \frac{dx}{dt}$$

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or $\dot{x} = f(t, x)$ where $x(t)$ is the unknown function.

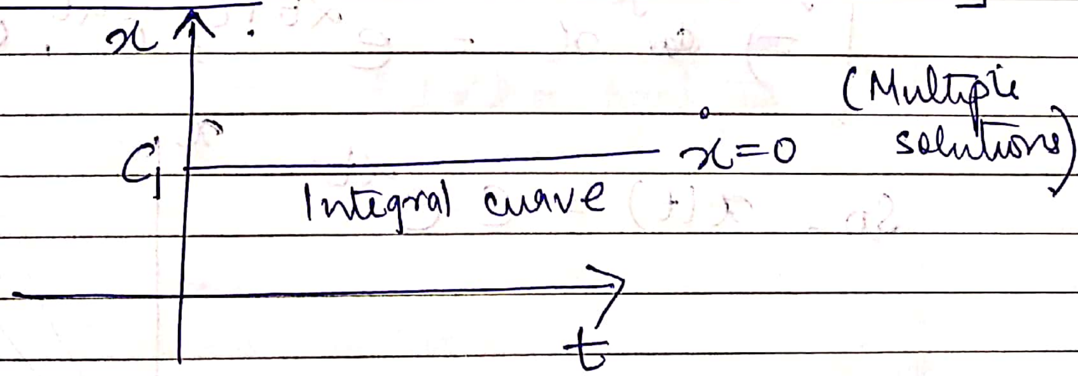
We wish to find a function x that satisfies the equation for all values of t

Result: A differentiable function $x = x(t)$ that is defined in an open interval I of the real line and satisfies $\dot{x} = f(t, x)$ for all t in I is called solution of $\dot{x} = f(t, x)$ in I. The graph of any solution is called an integral curve.

Examples

(i) $\boxed{\dot{x} = 0.}$ - [variable doesnot change over time]

Possible solution : $x(t) = C$ [constant]



(ii) $\boxed{\dot{x} = \alpha}$

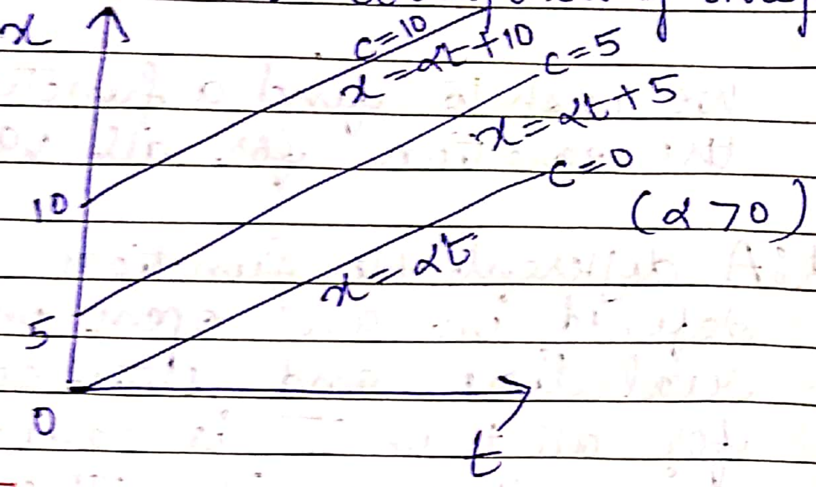
Time path of variable is such that rate of change of x per unit of time is constant.

Possible solution

$x(t) = \alpha t + C$

[Using integration
$\int \dot{x}\, dt = \int \alpha\, dt$
$x(t) = \alpha t + C$]

Depending on the value of C, there will be many solutions and corresponding integral curves.



(iii) $\boxed{\dot{x} = \alpha x}$

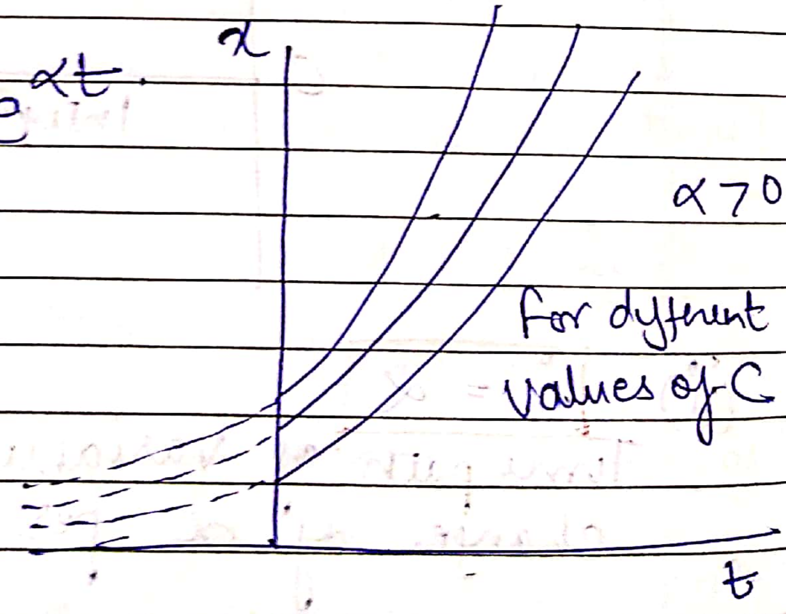
Possible solution

$$\dot{x} = \alpha x \quad \text{or} \quad \dot{x}/x = \alpha \Rightarrow \int \frac{\dot{x}}{x}\,dt = \int \alpha\,dt$$

$$\Rightarrow \log_e x = \alpha t + C$$

$$\Rightarrow x = e^{\alpha t + C} = e^{\alpha t} \cdot e^{C} = C e^{\alpha t}$$

So. $x(t) = C e^{\alpha t}$



The differential equations in which t does not appear explicitly are known as autonomous differential equations

We will be discussing more general cases later. Let's us first discuss, if we are given a function $x = \phi(t)$, How to check whether it is a solution of the differential equation $F(t, x, \dot{x}) = 0$ or $\dot{x} = f(t, x)$ or not. The following result will be applicable for the same.

Result: Given any ordinary differential equation of first order $F(t, x, \dot{x}) = 0$. where the domain of F is a subset of $\mathbb{R}^3$ and the range of F is a subset of $\mathbb{R}$ and given any function ϕ whose domain and range are subsets of $\mathbb{R}$ then $x = \phi(t)$ is a solution for $F(t, x, \dot{x}) = 0$ on a real interval I if and only if for all $t \in I$

$$F(t, \phi(t), \dot{\phi}(t)) = 0$$

Example

$$\dot{x} = x + t$$

or $F(t, x, \dot{x}) = \dot{x} - x - t = 0$

Given $\phi_1(t) = -t-1$; $\phi_2(t) = e^t - t - 1$

and $\phi_3(t) = e^t - 1$

Find which are the solutions for the differential equation.

Ans We know from the above result that any solution $\phi(t)$ must satisfy $F(t, \phi(t), \dot{\phi}(t)) = 0$.

(i) $\phi_1(t) = -t-1$

$\dot{\phi}_1(t) = -1$.

Putting in differential equation.

$F(t, \phi(t), \dot{\phi}(t)) = 0 \Rightarrow \dot{\phi}(t) - \phi(t) - t = 0$

$\Rightarrow -1 - (-t-1) - t = 0$

$\Rightarrow -1 + t + 1 - t = 0$

So $\phi_1(t)$ satisfies above differential equation. It is a solution.

(ii) $\phi_2(t) = e^t - t - 1$

$\dot{\phi}_2(t) = e^t - 1$

$F(t, \phi(t), \dot{\phi}(t)) = \dot{\phi}(t) - \phi(t) - t$

$= (e^t - 1) - (e^t - t - 1) - t$

$= 0$ So it is a solution

(iii) $\phi_3(t) = e^t - 1$

$\dot{\phi}_3(t) = e^t$

$$f(t, \phi(t), \dot{\phi}(t)) = \dot{\phi}(t) - \phi(t) - t$$
$$= e^t - (e^t - 1) - t$$
$$= 1 - t \neq 0 \text{ except } t = 1$$

So $\phi_3(t)$ is not a solution.

Initial Value Problem (IVP)

The set of all solutions of any differential equation is called the general solution of the equation. A first order differential equation usually have a general solution that depends on one constant. If we require the solution to pass through a given point in the tx plane, then the constant is determined uniquely.

So following is the IVP :

Solve
$$f(t, x, \dot{x}) = 0$$
$$x(t_0) = x_0,$$
$$x_0 \in \mathbb{R} \text{ and } t_0 \in \mathbb{R}.$$

For $x = \phi(t)$ on I to be solution then for all $t \in I$, $f(t, \phi(t), \dot{\phi}(t)) = 0$

$$\& \ \phi(t_0) = x_0$$

eg $\left.\begin{array}{l}\dot{x} = x \\ x(1) = 0.05\end{array}\right]$ IVP

This DE is of the form $\dot{x} = \alpha x$.
and we know it's solution is $\phi(t) = Ce^{\alpha t}$

Here $\alpha = 1$ so $\phi(t) = ce^{t}$
& $\phi(1) = 0.05$

So. $\phi(1) = ce^{1} = 0.05$

$\Rightarrow C = 0.05 e^{-1}$

So. $\boxed{x(t) = \phi(t) = 0.05\, e^{t-1}}$ solution of above IVP

- If the differential equation is of the form $\dot{x} = f(t)$ then

$\dot{x} = f(t) \Leftrightarrow \int \dot{x}\,dt = \int f(t)\,dt$.

or $x(t) = \int f(t)\,dt + C$.

eg $\dot{x} = t^2 - 1$ and $x(0) = 2$

$\int \dot{x}\,dt = \int (t^2 - 1)\,dt$

$x(t) = t^3/3 - t + C$

$x(0) = C = 2$ so $\boxed{x(t) = \frac{t^3}{3} - t + 2}$

First order Linear Differential Equations :

$$\dot{x} + a(t)x = b(t)$$

a and b are continuous functions of t in a certain interval. It is called linear because left hand side is a linear function of x and $\dot{x}$.

Solution (Integrating Factor Method)

Let us define IF as $I = e^{A(t)} = e^{\int a(t)dt}$

$$\dot{x} + a(t)x = b(t)$$

Multiply both sides by I.

$$[\dot{x} + a(t)x]\, e^{\int a(t)dt} = b(t)\, e^{\int a(t)dt}$$

$$\Leftrightarrow \dot{x} e^{\int a(t)dt} + a(t)x e^{\int a(t)dt} = b(t) e^{\int a(t)dt}$$

Now left hand side is simply the derivative of $x(t) e^{\int a(t)dt}$

$$\frac{d}{dt}\left[x e^{\int a(t)dt}\right] = \dot{x} e^{\int a(t)dt} + x e^{\int a(t)dt} a(t)$$

[as derivative of $e^{g(t)}$ is $e^{g(t)} g'(t)$]

$$\Rightarrow \frac{d}{dt}\left[x e^{\int a(t)dt}\right] = b(t) e^{\int a(t)dt}$$

Taking integral on both sides

$$\int \frac{d}{dt}\left[x\, e^{\int a(t)dt}\right] dt = \int b(t)\, e^{\int a(t)dt}\, dt$$

$$\Longleftrightarrow \quad x(t)\, e^{\int a(t)dt} = \int b(t)\, e^{\int a(t)dt}\, dt + C$$

$$\Longleftrightarrow \quad x(t) = e^{-\int a(t)dt}\left[\int b(t)\, e^{\int a(t)dt}\, dt + C\right]$$

So. $\dot{x} + a(t)x = b(t) \Longleftrightarrow$

(1) $$x(t) = e^{-\int a(t)dt}\left[\int b(t)\, e^{\int a(t)dt}\, dt + C\right]$$

Sub E. Cases :

(a) If $a(t) = a \quad \forall t$

then DE is $\dot{x} + ax = b(t)$

From (1). $e^{\int a\, dt} = e^{at}$

So. $x(t) = e^{-at}\left[C + \int e^{at} b(t)\, dt\right]$

(b) If $a(t) = a$ and $b(t) = b \quad \forall t$

then DE is $\dot{x} + ax = b$.

From (1) $e^{-\int a(t)\, dt} = e^{-\int a\, dt} = e^{-at}$

$\int e^{at}\, b\, dt = (b/a)\, e^{at}$

So $x(t) = Ce^{-at} + b/a$

When $X(t_0) = X_0$ is given, we can find the value of C in (1)

Consider the solution

$$x(t) = e^{-\int a(t)dt}\left[C + \int b(t) e^{\int a(t)dt} dt\right]$$

Let f be a function such that $\dot{f} = b(t)e^{A(t)}$ where $A(t) = \int a(t)dt$ and so $A(t) - A(s)$

Solution can be written as

$$x(t) = C e^{-A(t)} + e^{-A(t)} f(t) \quad — (2)$$

Let $t = t_0$ and solve for C to get

$$C = x(t_0) e^{A(t_0)} - f(t_0)$$

Put value of C in equation (2)

So. $x(t) = x(t_0) e^{-(A(t) - A(t_0))} + e^{-A(t)}[f(t) - f(t_0)]$

By definition of $f(t)$

$$f(t) - f(t_0) = \int_{t_0}^{t} b(s) e^{A(s)} ds$$

So. $e^{-A(t)}[f(t) - f(t_0)] = e^{-A(t)} \int_{t_0}^{t} b(s) e^{A(s)} ds$

$$= \int_{t_0}^{t} b(s) e^{-(A(t) - A(s))} ds$$

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We can include $e^{-A(t)}$ in the integrand because we are integrating w.r.t s.

So, when $x(t_0) = x_0$

$$\dot{x} + a(t)x = b(t) \iff$$

$$x(t) = x_0 e^{-\int_{t_0}^{t} a(r)dr} + \int_{t_0}^{t} b(s) e^{-\int_{s}^{t} a(r)dr} ds$$

Examples

① Find the solution on $(-\infty, \infty)$ for the following IVP

$$\dot{x} - 3x - e^{2t} = 0 \quad ; x(0) = 0$$

Solution IF $= e^{\int -3\,dt} = e^{-3t}$

$$\dot{x} - 3x = e^{2t} \qquad a(t) = -3$$
$$b(t) = e^{2t}.$$

Multiplying IF on both sides

$$(\dot{x} - 3x)e^{-3t} = e^{2t} \cdot e^{-3t} = e^{-t}$$

$$\Rightarrow \frac{d}{dt}\left[x(t)e^{-3t}\right] = e^{-t}$$

Taking integral on both sides

$$\int_0^t \frac{d}{dt}\left[x(t) e^{-3t}\right] = \int_0^t e^{-t} dt$$

$$\left[x(t)e^{-3t}\right]_0^t = \left[\frac{e^{-t}}{-1}\right]_0^t$$

$$x(t)e^{-3t} - x(0)e^{0} = -e^{-t} + 1$$

$$x(t)e^{-3t} - 0 = -e^{-t} + 1$$

$$\boxed{x(t) = e^{3t} - e^{2t}}$$ solution.

To show it's a solution.

$$F(t, \phi(t), \dot{\phi}(t)) = 0.$$

our DE $\quad \dot{\phi}(t) - 3\phi(t) - e^{2t} = 0$

$$\phi(t) = e^{3t} - e^{2t}$$

$$\dot{\phi}(t) = 3e^{3t} - 2e^{2t}$$

Put in DE $\Rightarrow$ $3e^{3t} - 2e^{2t} - 3(e^{3t} - e^{2t}) - e^{2t}$

$$= 0.$$

So $x(t) = \phi(t) = e^{3t} - e^{2t}$ is a solution of given DE.

② Find a solution on $(0,\infty)$ for the following IVP:

$$t\dot{x} - 2x - t^5 = 0 \quad ; \quad x(1) = 1$$

Solution: To use the method of I.F, coefficient of $\dot{x}$ must be 1

So. $\dot{x} - \frac{2x}{t} - t^4 = 0$

$$\dot{x} - \frac{2x}{t} = t^4 \quad ; \quad a(t) = -2/t$$

$$b(t) = t^4$$

$$I.f = e^{-\int -2/t \, dt} = e^{-2\log t} = e^{\log t^{-2}}$$

$$I.f = 1/t^2$$

Multiplying I.f on both sides

$$\left[\dot{x} - \frac{2x}{t}\right] 1/t^2 = t^4 \times \frac{1}{t^2}$$

$$\Rightarrow \frac{d}{dt}\left[x(t)\, 1/t^2\right] = t^2.$$

Taking integral on both sides

$$\int_1^t \frac{d}{dt}\left[x(t) \frac{1}{t^2}\right] dt = \int_1^t t^2 \, dt.$$

$$\left[x(t)\, 1/t^2\right]_1^t = \left[t^3/3\right]_1^t$$

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$$x(t)\,{}^{1}/_{t^2} - x(1) = t^3/_3 - 1/_3$$

$$x(t)\frac{1}{t^2} - 1 = t^3/_3 - \frac{1}{3}$$

$$\boxed{x(t) = t^5/_3 + 2t^2/_3}$$

We can also check it's a solution by putting it back in DE.

$$t\dot{x} - 2x - t^5 = 0 \longrightarrow DE$$

$$\dot{x} = 5t^4/_3 + 4t/_3$$

$$t\left[5t^4/_3 + 4t/_3\right] - 2\left[t^5/_3 + 2t^2/_3\right] - t^5$$

$$= 5t^5/_3 + 4t^2/_3 - 2t^5/_3 - 4t^2/_3 - t^5 = 0$$

So our solution satisfies DE.

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Stability and Phase diagrams

We are often interested not in the exact form of the solution of a differential equation, but only in the qualitative properties of this solution. One of the most important properties of a differential equation is whether it has any equilibrium states. They correspond to solutions of the equations that do not change over time. We would also like to know whether an equilibrium state is stable. For instance, the rest position of a pendulum (hanging downward and motionless) is stable; if it is slightly disturbed while in this position, it will swing around it and gradually approach the equilibrium state of rest.

Let us consider the autonomous differential equation.

$$\dot{x} = F(x)$$

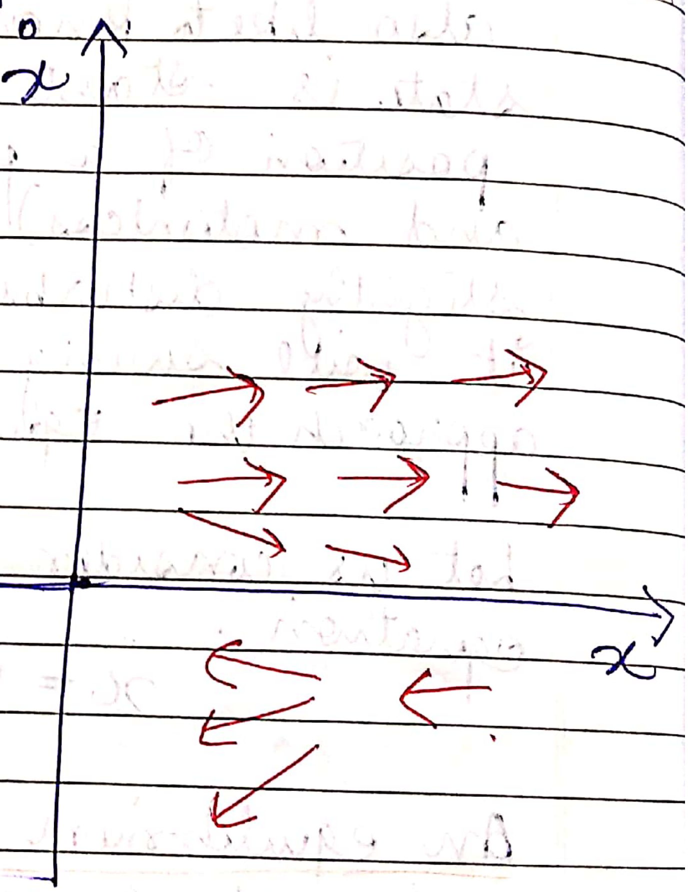
An equilibrium or a stationary state for above equation is a value of x for which $F(x)=0$ [because if $F(x)=0$ then $\dot{x}=0$ so that the value of x does not change]

Let $F(x)=0$ when $x=a$ so $F(a)=0$.
Then $x(t) = a$ (for all t) is a solution of the equation.
If $x(t_0)=a$ for some value t_0 of t then $x(t)$ is equal to a for all t

To examine the stability properties of the equilibrium states, it is useful to study its Phase diagram.

Phase diagram is the geometric representation of differential equation $\dot{x} = F(x)$ in the $x\dot{x}$ plane.

or Take $y = \dot{x}$ so we have $y = F(x)$ in the xy plane. The graph representing the function $F(x)$ is called Phase line.



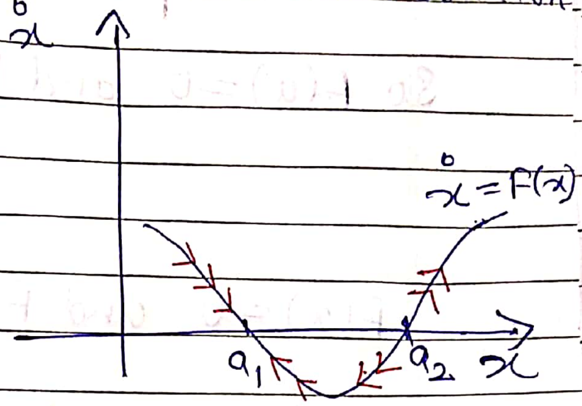
Note:

① If we are anywhere above the horizontal axis $\frac{dx}{dt}$ or $\dot{x} > 0$ which means x must be increasing over time so the pt $(x, \dot{x})$ on the phase line moves from left to right

② If we are at a point on the graph below the x axis then $\dot{x} < 0$ and $x(t)$ decreases with t, so we move from right to left.

Let $\dot{x} = f(x)$ has two equilibrium states a_1 and a_2. If we are at one of these states, then we will remain there.

- If $x(t)$ is close to a_1 but not equal to a_1 then $x(t)$ will approach a_1 as t increases.



[If we are on the left side of a_1, there the phase line is above the x axis, so x will increase over time so eventually it will reach a_1. Similarly if we are on the right side of a_1 $[x > a_1]$, phase line is below the x axis, so x decreases over time $[dx/dt < 0]$ so eventually it will reach a_1].

- If $x(t)$ is close to a_2, but not equal to a_2, then $x(t)$ will start to move away from a_2 as t increases.

[as if $x < a_2$, $\frac{dx}{dt} < 0$ so x will decrease over time, move further away from a_2. Similarly, $x > a_2$, $\frac{dx}{dt} > 0$ so x will increase over time, move further away]

- So a_1 is a stable equilibrium state and a_2 is unstable equilibrium state

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- Note at the stable pt a_1, the graph of $\dot{x} = f(x)$ has a negative slope where the slope is positive at a_2

So $f(a) = 0$ and $f'(a) < 0 \Rightarrow$ 'a' is a stable equilibrium state for $\dot{x} = f(x)$.

$f(a) = 0$ and $f'(a) > 0 \Rightarrow$ 'a' is an unstable equilibrium state for $\dot{x} = f(x)$

Special case

(1) 1st DE $\dot{x} + ax = b$. or $\dot{x} = b - ax = f(x)$

For Equilibrium $\dot{x} = 0 \Rightarrow b - ax = 0 \Rightarrow$
$x = b/a$.

$f'(x) = -a < 0$ if $a > 0$.
> 0 if $a < 0$

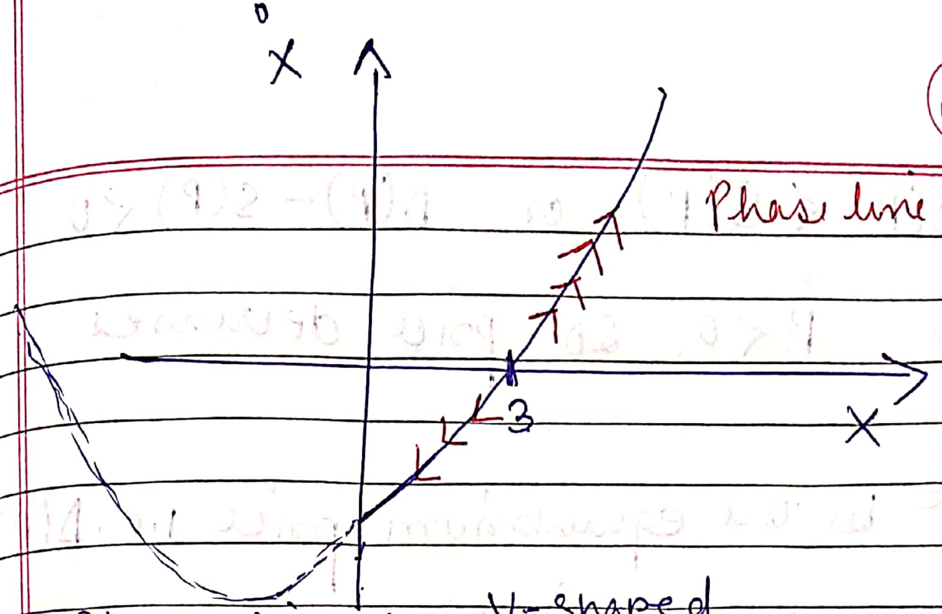
So $x = b/a$ is a stable eq^m of $a > 0$

and unstable eq^m if $a < 0$.

(2) $\dot{x} = (x+1)^2 - 16$ $(x \geq 0)$

Eq$^m \Rightarrow (x+1)^2 - 16 = 0 \Rightarrow \boxed{x = 3}$ or $x = -5$
Ruled out

$f'(x) = 2(x+1)$
$f'(3) = 2(3+1) = 8 > 0$ unstable eq^m.



Phase line is a V-shaped parabola with center at $(-1, -16)$

Dynamics of Market Price

$Q_d = \alpha - \beta P \quad (\alpha, \beta > 0)$
$Q_s = -\gamma + \delta P \quad (\gamma, \delta > 0)$

$P^* = \dfrac{\alpha + \gamma}{\beta + \delta}$; If $P(0)$ [initial price] $= P^*$

then market will clearly be in equilibrium already and no dynamic analysis will be needed.

Normally $P(0) \neq P^*$

Let the change in price over time is a function of excess demand $D(P) - S(P)$

$$\dot{P} = H\left(D(P) - S(P)\right)$$

Let $H(0) = 0$ and $\dot{H} > 0$. So if $D(P) = S(P)$, $\dot{P} = 0$, and if $D(P) > S(P)$ So ~~D(P) > S(P)~~

Excess demand over supply: $D(P) - S(P) > 0$ then $\dot{P} > 0$ so Price increases

Excess Supply [If $D(P) < S(P)$ or $D(P) - S(P) < 0$
then $\dot{P} < 0$ so price decreases

Let P^e be the equilibrium price i.e. $D(P) = S(P)$
So $H(D(P^e) - S(P^e)) = 0$ and hence

$$D(P^e) - S(P^e) = 0 \text{ [demand = Supply]}$$

Now $F(P) = H(D(P) - S(P))$

$$F'(P) = H'(D(P) - S(P)) [D'(P) - S'(P)]$$

Since $H' > 0$, so for stable equilibrium
$F'(P^e) < 0$ so $D'(P^e) - S'(P^e) < 0$

$$\Rightarrow \boxed{D'(P^e) < S'(P^e)}$$

Usually for ordinary goods, demand curve is downward sloping and Supply curve is upward sloping so
$D'(P) < 0$ and $S'(P) > 0$ so above

Stability condition is automatically satisfied.